Differential Geometry III: Riemannian geometry

G. Moschidis
28 Feb. 2025

2.1 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold and $\gamma : [a,b] \to \mathcal{M}$ a curve of class C^1. Recall that the *length* of γ is defined as

$$\ell(\gamma) \doteq \int_a^b \|\dot{\gamma}(t)\| \, dt.$$

We will also define the *energy* of γ by the relation

$$\mathcal{E}(\gamma) \doteq \int_a^b \|\dot{\gamma}(t)\|^2 \, dt.$$

(a) Show that $\ell(\gamma)$ is invariant under reparametrizations of γ (i.e. that it is the same for the curves γ and $\gamma \circ h$, where $h : [a,b] \to [a',b']$ is any C^1 bijection). Is the energy also similarly invariant under reparametrizations?

(b) Show that

$$\big(\ell(\gamma)\big)^2 \leqslant (b-a)\mathcal{E}(\gamma).$$

When does equality hold above?

2.2 Let $(\mathcal{M}, g)$ be a smooth connected Riemannian manifold. For any $p, q \in \mathcal{M}$, let $\mathcal{C}_{p,q}$ be the set of all *piecewise* C^1 curves $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Recall that the Riemannian distance function $d_g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is defined by the formula

$$d_g(p,q) = \inf \Big\{ \ell(\gamma) \,|\, \gamma \in \mathcal{C}_{p,q} \Big\}$$

where $\ell(\gamma)$ is the length of γ with respect to the Riemannian metric g. Show that $(\mathcal{M}, d_g)$ is indeed a metric space.
Hint: Showing that $p = q$ if $d_g(p,q) = 0$ is a bit tricky; you can try establishing first the following intermediate step: Show that, for any $p \in \mathcal{M}$ and any local coordinate chart $\phi : \mathcal{U} \to \mathbb{R}^n$ (where $n = dim\mathcal{M}$) on a neighorhood $\mathcal{U}$ of p, there exist $\delta, \eta > 0$ such that for any point $q \in \mathcal{V}_\delta \doteq \phi^{-1}(B_\delta(\phi(p)))$ (where $B_\delta(\phi(p))$ is the Euclidean open ball in $\mathbb{R}^n$ of radius δ centered at $\phi(p)$) and any $X \in T_q\mathcal{M}$, we have

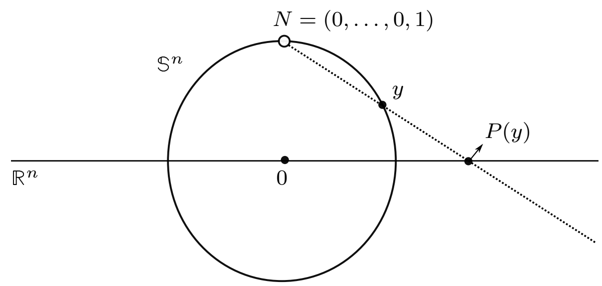
$$g(X,X) \geqslant \eta \sum_{i=1}^n (X^i)^2.$$

Deduce from this that, for any piecewise C^1 curve γ in $\mathcal{M}$ passing through p and contained in $\mathcal{V}_\delta$, we can estimate

$$\ell(\gamma) \geqslant \sqrt{\eta} \cdot \ell_E(\phi \circ \gamma),$$

where ℓ_E denotes the Euclidean length of a curve in $\mathbb{R}^n$.

2.3 For $n \geqslant 1$, let $N = (0, 0, \ldots, 0, 1)$ be the north pole of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Let $\mathcal{P} : \mathbb{S}^n \setminus N \to \mathbb{R}^n$ be the stereographic projection via N onto the hyperplane $x^{n+1} = 0$, that is to say, for any $y \in \mathbb{S}^n \setminus N$, $P(y) = (y_1, \ldots y_n)$ is defined so that the point $(y_1, \ldots, y_n, 0)$ belongs to the straight line in $\mathbb{R}^{n+1}$ connecting N to y.



(a) Show that the round metric $g_{\mathbb{S}^n}$, i.e. the metric induced on $\mathbb{S}^n$ from the Euclidean metric on $\mathbb{R}^{n+1}$, takes the following form in the coordinate chart determined by P on $\mathbb{S}^n \setminus N$:

$$g_{\mathbb{S}^n} = \frac{4}{\big(1 + \|y\|^2\big)^2} \Big(dy_1^2 + dy_2^2 + \ldots + dy_n^2 \Big)$$

(b) Show that the map $P : (\mathbb{S}^n, g_{\mathbb{S}^n}) \to (\mathbb{R}^n, g_E)$ (where g_E is the Euclidean metric on $\mathbb{R}^n$) is conformal.

(c) Consider the map $F : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ given by

$$F(x) = \frac{x}{\|x\|^2}.$$

Show that, in the coordinate chart above, the map F defines an isometry of $\mathbb{S}^n \setminus \{N, S\}$ to itself, where $S = (0, 0, \dots, 0, -1)$ is the south pole of $\mathbb{S}^n$. Does this map extend as an isometry on the whole of $\mathbb{S}^n$?

2.4 Let $\mathcal{M}$ be a smooth manifold of dimension n.

(a) Let V be a smooth vector field on $\mathcal{M}$ such that $V(p) \neq 0$ for some $p \in \mathcal{M}$. Show that there exists an open neighborhood $\mathcal{U}$ of p and a local coordinate system $(y^1, \dots, y^n)$ on $\mathcal{U}$ such that $V = \frac{\partial}{\partial y^1}$ on $\mathcal{U}$.
Hint: You can use the following result from classical ODE theory as a black box for this exercise:
Theorem: Let S be a smooth hypersurface of $\mathbb{R}^n$ and X a smooth vector field on $\mathbb{R}^n$ such that X is transversal to S and non-vanishing in a neighborhood of S. Then, for any $f \in C^\infty(\mathbb{R}^n)$, $g \in C^\infty(S)$, the initial value problem

$$\begin{cases} X(u) = f, \\ u|_S = g \end{cases}$$

has a unique smooth solution u defined on the open subset of $\mathbb{R}^n$ covered by integral curves of X emanating from S.

(b) For V as above, let W be another smooth vector field on $\mathcal{M}$ such that $W(p) \neq 0$ and $W(p) \neq V(p)$. Is it always true that we can find a local coordinate system $(y^1, \dots, y^n)$ on a neighborhood $\mathcal{U}$ of p as before such that $V = \frac{\partial}{\partial y^1}$ and $W = \frac{\partial}{\partial y^2}$ on $\mathcal{U}$? (*Hint: Consider the commutator* $[V, W](f) \doteq V(W(f)) - W(V(f))$ *for a suitable function* $f \in C^\infty(\mathcal{M})$.)

(c) Let ω be an 1-form on $\mathcal{M}$ such that $\omega(p) \neq 0$ for some $p \in \mathcal{M}$. Does there always exist a local coordinate system $(y^1, \dots, y^n)$ on a neighborhood $\mathcal{U}$ of p such that $\omega = dy^1$ in $\mathcal{U}$?

***2.5** Recall that the real projective space $\mathbb{P}^n(\mathbb{R})$ is the space of straight lines in $\mathbb{R}^{n+1}$ passing through the origin. In other words, $\mathbb{P}^n(\mathbb{R})$ is the space of equivalence classes $[x] = \{y \in \mathbb{R}^{n+1} \setminus \{0\} : y = \lambda x,\ \lambda \neq 0\}$ in $\mathbb{R}^{n+1} \setminus \{0\}$.

The space $\mathbb{P}^n(\mathbb{R})$ has a natural manifold structure; a smooth atlas on $\mathbb{P}^n(\mathbb{R})$ is given by $\{\mathcal{U}^{(k)}, \phi_k\}_{k=1}^{n+1}$, where

$$\mathcal{U}^{(k)} = \big\{[(x^1, \dots, x^{n+1})] \in \mathbb{P}^n(\mathbb{R}) :\ x^k \neq 0\big\}$$

and the maps $\phi_k : \mathcal{U}^{(k)} \to \mathbb{R}^{n+1}$ are homeomorphisms defined by the following relation for ϕ_k^{-1}:

$$\phi_k^{-1}(y^1, \dots, y^n) = \big[(x^1, \dots, x^{n+1})\big] \quad \text{with } x^j = \begin{cases} y^j \text{ for } j \leqslant k-1, \\ 1 \text{ for } j = k, \\ y^{j-1} \text{ for } j \geqslant k+1. \end{cases}$$

- Show that the transition maps $\phi_i \circ \phi_j^{-1}$, $i \neq j \in \{1, \dots, n+1\}$, are of class C^∞ (in fact, real analytic) on their domain of definition.

Let us equip $\mathbb{P}^n(\mathbb{R})$ with the standard projective metric $g_{\mathbb{P}^n}$; the components of the matrix for the metric in each of the coordinate charts associated to ϕ_k above take the following form:

$$(g_{\mathbb{P}^n})_{ij} = \frac{1}{1 + \|y\|^2}\Big(\delta_{ij} - \frac{y^i y^j}{1 + \|y\|^2}\Big).$$

- Show that the natural map $\mathcal{F} : (\mathbb{S}^n, g_{\mathbb{S}^n}) \to (\mathbb{P}^n, g_{\mathbb{P}^n})$ defined by $\mathcal{F}(x) = [x]$ (i.e. sending each point on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ to the corresponding straight line connecting it to 0) is a local isometry.
- Does there exist a global isometry between $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $(\mathbb{P}^n, g_{\mathbb{P}^n})$? (*Hint: Compare the volumes of* $(\mathbb{S}^n, g_{\mathbb{S}^n})$ *and* $(\mathbb{P}^n, g_{\mathbb{P}^n})$.)